

# ON A THEOREM OF SCHWICK

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**ABSTRACT.** Let  $\mathcal{D}$  be a domain,  $n, k$  be positive integers and  $n \geq k + 3$ . Let  $\mathcal{F}$  be a family of functions meromorphic in  $\mathcal{D}$ . If each  $f \in \mathcal{F}$  satisfies  $(f^n)^{(k)}(z) \neq 1$  for  $z \in \mathcal{D}$ , then  $\mathcal{F}$  is a normal family. This result was proved by Schwick [10], in this paper we extend this theorem.

## 1. INTRODUCTION AND MAIN RESULTS

We denote the complex plane by  $\mathbb{C}$ , and the unit disk  $\{z \in \mathbb{C} : |z| < 1\}$  by  $\Delta$ . In 1989, Schwick [10] proved a normality criterion which states that: *For positive integers  $k, n \geq k+3$ , let  $\mathcal{F}$  be a family of functions meromorphic in  $\mathcal{D}$ . If each  $f \in \mathcal{F}$  satisfies  $(f^n)^{(k)}(z) \neq 1$  for  $z \in \mathcal{D}$ , then  $\mathcal{F}$  is a normal family.* This result holds good for holomorphic functions with the case  $n \geq k + 1$ . The following theorem is a result of Wang and Fang [12]. The proof was omitted in that article, here we give a proof of this result and extend this theorem.

**Theorem 1.1.** *Let  $n, k$  be positive integers and  $n \geq k + 1$  and  $\mathcal{D}$  be a domain in  $\mathbb{C}$ . Let  $\mathcal{F}$  be a family of functions meromorphic on  $\mathcal{D}$ . If each  $f \in \mathcal{F}$  satisfies  $(f^n)^{(k)}(z) \neq 1$  for  $z \in \mathcal{D}$ , then  $\mathcal{F}$  is a normal family.*

It is natural to ask what can happen if we have a solution of  $(f^n)^{(k)} - 1$ . For this question we can extend Theorem 1.1 for the case  $k \geq 1$  in the following manner.

**Theorem 1.2.** *Let  $n, k$  be positive integers and  $n \geq k + 2$  and  $\mathcal{D}$  be a domain in  $\mathbb{C}$ . Let  $\mathcal{F}$  be a family of functions meromorphic on  $\mathcal{D}$ . If for each function  $f \in \mathcal{F}$ ,  $(f^n)^{(k)}(z) - 1$  has at most one zero ignoring multiplicity (IM) in  $\mathcal{D}$ , then  $\mathcal{F}$  is a normal family.*

In this paper, we use the following standard notations of value distribution theory,

$$T(r, f); m(r, f); N(r, f); \overline{N}(r, f), \dots$$

We denote  $S(r, f)$  any function satisfying

$$S(r, f) = o\{T(r, f)\}, \text{ as } r \rightarrow +\infty,$$

possibly outside of a set with finite measure.

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## 2. PRELIMINARY RESULTS

In order to prove our results we need the following Lemmas.

**Lemma 2.1.** { [15], p. 216; [15], p. 814 } (Zalcman's lemma)

Let  $\mathcal{F}$  be a family of meromorphic functions in the unit disk  $\Delta$ , with the property that for every function  $f \in \mathcal{F}$ , the zeros of  $f$  are of multiplicity at least  $l$  and the poles of  $f$  are of multiplicity at least  $k$ . If  $\mathcal{F}$  is not normal at  $z_0$  in  $\Delta$ , then for  $-l < \alpha < k$ , there exist

- (1) a sequence of complex numbers  $z_n \rightarrow z_0$ ,  $|z_n| < r < 1$ ,
- (2) a sequence of functions  $f_n \in \mathcal{F}$ ,
- (3) a sequence of positive numbers  $\rho_n \rightarrow 0$ ,

such that  $g_n(\zeta) = \rho_n^\alpha f_n(z_n + \rho_n \zeta)$  converges to a non-constant meromorphic function  $g$  on  $\mathbb{C}$  with  $g^\#(\zeta) \leq g^\#(0) = 1$ . Moreover  $g$  is of order at most two.

**Lemma 2.2.** { [17], Lemma 2.5, Lemma 2.6; [18], Lemma 2.2 } Let  $R = \frac{P}{Q}$  be a rational function and  $Q$  be non-constant. Then  $(R^{(k)})_\infty \leq (R)_\infty - k$ , where  $k$  is a positive integer,  $(R)_\infty = \deg(P) - \deg(Q)$  and  $\deg(P)$  denotes the degree of  $P$ .

**Lemma 2.3.** [17] Let  $R = a_m z^m + \dots + a_1 z + a_0 + \frac{P}{B}$ , where  $a_0, a_1, \dots, a_{m-1}, a_m (\neq 0)$  are constants,  $m$  is a positive integer and  $P, B$  are polynomials with  $\deg(P) < \deg(B)$ . If  $k \leq m$ , then  $(R^{(k)})_\infty = (R)_\infty - k$ ,

**Lemma 2.4.** Let  $k, n$  is two positive integer and  $n \geq k + 1$ . Let  $f$  be a non-constant rational function then  $(f^n)^{(k)} - b$  has a root for all nonzero complex numbers  $b$ .

*Proof.* Suppose  $(f^n)^{(k)} - b$  has no root. First we suppose  $f$  is a non-constant polynomial of degree  $d \geq 1$ , then  $(f^n)^{(k)} - b$  is a polynomial of degree  $nd - k \geq 1$ . Thus by fundamental theorem of algebra  $(f^n)^{(k)} - b$  has a solution.

Again, let  $f$  is a non-polynomial rational function. We set

$$(2.1) \quad f(z) = A \frac{(z - \alpha_1)^{m_1} (z - \alpha_2)^{m_2} \dots (z - \alpha_s)^{m_s}}{(z - \beta_1)^{n_1} (z - \beta_2)^{n_2} \dots (z - \beta_t)^{n_t}},$$

where  $A$  is a nonzero constant and  $m_1, m_2, \dots, m_s, n_1, n_2, \dots, n_t$  are positive integers. We denote

$$(2.2) \quad (f^n)^{(k)}(z) = \frac{(z - \alpha_1)^{nm_1 - k} (z - \alpha_2)^{nm_2 - k} \dots (z - \alpha_s)^{nm_s - k} g(z)}{(z - \beta_1)^{nn_1 + k} (z - \beta_2)^{nn_2 + k} \dots (z - \beta_t)^{nn_t + k}} = \frac{p}{q},$$

where  $g(z)$  is a polynomial and  $\deg(g) \leq k(s + t - 1)$ . Suppose  $(f^n)^{(k)}(z) \neq b$ , then

$$(2.3) \quad (f^n)^{(k)}(z) = b + \frac{B}{(z - \beta_1)^{nn_1 + k} (z - \beta_2)^{nn_2 + k} \dots (z - \beta_t)^{nn_t + k}} = \frac{p}{q}$$

from (2.2) and (2.3)  $N + kt = \deg(q) = \deg(p) = M - ks + \deg(g) \leq M - ks + k(s + t - 1) = M + kt - k$ . This gives  $M - N \geq k$  i.e.  $n(\sum_{s=1}^i m_i - \sum_{t=1}^i n_i) \geq k$ . This implies  $(\sum_{i=1}^s m_i - \sum_{i=1}^t n_i) > 1$ . So  $(f)_\infty > 1$  hence  $(f^n)_\infty > n$ . Therefore we can express  $f^n$  as follows

$$f^n(z) = a_m z^m + \dots + a_1 z + a_0 + \frac{P}{B},$$

where  $a_0, a_1, \dots, a_{m-1}, a_m (\neq 0)$  are constants,  $m \geq n$  is an integer,  $P$  and  $B$  are polynomials with  $\deg(P) < \deg(B)$ . Since  $m > k$ , then by 2.3 we get

$$((f^n)^{(k)})_\infty = (f^n)_\infty - k > n - k \geq 1,$$

which contradicts the fact that  $\deg(p) = \deg(q)$ . Thus  $(f^n)^{(k)}(z) - b$  has a solution in  $\mathbb{C}$ .  $\square$

**Lemma 2.5.** [8] *Let  $n, k$  be positive integers such that  $n \geq k + 2$  and  $a \neq 0$  be a finite complex number, and  $f$  be a non-constant rational meromorphic function, then  $(f^n)^{(k)} - a$  has at least two distinct zeros.*

**Lemma 2.6.** { [13] P.38} *Let  $f(z)$  be a transcendental meromorphic function on  $\mathbb{C}$ , then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f)$$

*for every positive integer  $k$ .*

**Lemma 2.7.** *Let  $f$  be a non-constant meromorphic function on  $\mathbb{C}$ . Then*

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f)$$

*Proof.*

$$\begin{aligned} T(r, f^{(k)}) &= N(r, f^{(k)}) + m(r, f^{(k)}) \\ &\leq N(r, f) + k\overline{N}(r, f) + m(r, f) + m(r, \frac{f^{(k)}}{f}) \\ &\leq T(r, f) + k\overline{N}(r, f) + S(r, f) \end{aligned}$$

$\square$

**Lemma 2.8.** [6] *Let  $f(z)$  be a transcendental meromorphic function. Then for each positive number  $\epsilon$  and each positive integer  $k$ , we have*

$$k\overline{N}(r, f) \leq N(r, \frac{1}{f^{(k)}}) + N(r, f) + \epsilon T(r, f) + S(r, f).$$

**Lemma 2.9.** { [2] Corollary 3.} *If a meromorphic function of finite order  $\rho$  has only finitely many critical values, then it has at most  $2\rho$  asymptotic values.*

**Lemma 2.10.** [3] *Let  $g(z)$  be a transcendental meromorphic function and suppose that  $g(0) \neq \infty$  and the set of finite critical and asymptotic values of  $g(z)$  is bounded. then there exists  $R > 0$  such that*

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R},$$

for all  $z \in \mathbb{C} \setminus \{0\}$  which are not poles of  $g(z)$ .

**Lemma 2.11.** [11] *If  $f$  is a transcendental meromorphic function and  $k$  be a positive integer, then, for every positive number  $\epsilon$ ,*

$$(k-2)\overline{N}(r, f) + N(r, \frac{1}{f}) \leq 2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) + \epsilon T(r, f) + S(r, f).$$

The following lemma was proved by Bergweiler [2] and Wang [12] independently. Here we are giving another proof of this lemma.

**Lemma 2.12.** *Let  $f(z)$  be a transcendental meromorphic function with finite order. Let  $k, n$  be two positive integers such that  $n \geq k+1$ , then  $(f^n)^{(k)} - b$  has infinitely many zeros for all  $b \in \mathbb{C} \setminus \{0\}$ .*

*Proof.* Suppose on the contrary that  $(f^n)^{(k)}$  assumes the value  $b$  only finitely many times. Then

$$(2.4) \quad N(r, \frac{1}{(f^n)^{(k)} - b}) = O(\log r) = S(r, f).$$

By Nevanlinna's First Fundamental Theorem and Lemma 2.6 and Lemma 2.7

$$\begin{aligned} m(r, \frac{1}{f^n}) + m(r, \frac{1}{(f^n)^{(k)} - b}) &\leq m(r, \frac{(f^n)^{(k)}}{f^n}) + m(r, \frac{1}{(f^n)^{(k)}}) + m(r, \frac{1}{(f^n)^{(k)} - b}) \\ &\leq m(r, \frac{1}{(f^n)^{(k)}}) + \frac{1}{(f^n)^{(k)} - b} + S(r, f^n) \\ &\leq m(r, \frac{1}{(f^n)^{(k+1)}}) + m(r, \frac{(f^n)^{(k+1)}}{(f^n)^{(k)}} + \frac{(f^n)^{(k+1)}}{(f^n)^{(k)} - b}) + S(r, f^n) \\ &\leq m(r, \frac{1}{(f^n)^{(k+1)}}) + S(r, f^n) \\ &\leq T(r, (f^n)^{(k+1)}) - N(r, \frac{1}{(f^n)^{(k+1)}}) + S(r, f) \\ (2.5) \quad &\leq T(r, (f^n)^{(k)}) + \overline{N}(r, f^n) - N(r, \frac{1}{(f^n)^{(k+1)}}) + S(r, f^n). \end{aligned}$$

Together with Nevanlinna's First Fundamental Theorem this yields

$$\begin{aligned} T(r, f^n) &\leq \overline{N}(r, f^n) + N(r, \frac{1}{f^n}) \\ (2.6) \quad &+ N(r, \frac{1}{(f^n)^{(k)} - b}) - N(r, \frac{1}{(f^n)^{(k+1)}}) + S(r, f^n). \end{aligned}$$

First, we consider the case when  $k \geq 2$ , then By Lemma 2.11, for every  $\epsilon > 0$ , we have

$$\begin{aligned}
 (2.7) \quad & \overline{N}(r, f^n) + N(r, \frac{1}{f^n}) \\
 & \leq (k-1)\overline{N}(r, \frac{1}{f^n}) + N(r, \frac{1}{f^n}) \\
 & \leq 2\overline{N}(r, \frac{1}{f^n}) + N(r, \frac{1}{(f^n)^{(k+1)}}) + \epsilon T(r, f^n) + S(r, f^n).
 \end{aligned}$$

From (2.6) and (2.7), and using the fact that zeros of  $f^n$  has multiplicity at least 3 in this case, we get

$$\begin{aligned}
 (2.8) \quad & T(r, f^n) \leq 2\overline{N}(r, \frac{1}{f^n}) + N(r, \frac{1}{(f^n)^{(k)} - b}) + \epsilon T(r, f^n) + S(r, f^n) \\
 & \leq \frac{2}{3}N(r, \frac{1}{f^n}) + N(r, \frac{1}{(f^n)^{(k)} - b}) + \epsilon T(r, f^n) + S(r, f^n) \\
 & \leq \frac{2}{3}T(r, \frac{1}{f^n}) + N(r, \frac{1}{(f^n)^{(k)} - b}) + \epsilon T(r, f^n) + S(r, f^n) \\
 & \leq (\frac{2}{3} + \epsilon)T(r, f^n) + N(r, \frac{1}{(f^n)^{(k)} - b}) + S(r, f^n).
 \end{aligned}$$

Now, taking  $\epsilon = \frac{1}{6}$ , from (2.4) and (2.8), we obtain

$$T(r, f^n) \leq 6N(r, \frac{1}{(f^n)^{(k)} - b}) + S(r, f^n) = S(r, f^n),$$

which contradicts the fact that  $f$  is a transcendental meromorphic function. Thus, Lemma 2.11 is proved for the case  $k \geq 2$ .

Now, for the case  $k = 1$ , we use the method of Fang [5]. We first consider that  $f(z)$  has only finitely many zeros so is  $f^n(z)$  has only finitely many zeros *i.e.*  $N(r, \frac{1}{f^n}) = S(r, f^n)$ . and invoke the Lemma 2.8 and combine it with (2.6), we have

$$\begin{aligned}
 T(r, f^n) & \leq \overline{N}(r, f^n) + N(r, \frac{1}{f^n}) \\
 & \quad + N(r, \frac{1}{(f^n)' - b}) - N(r, \frac{1}{(f^n)''}) + S(r, f^n) \\
 & \leq \frac{1}{2}N(r, f^n) + N(r, \frac{1}{f^n}) + N(r, \frac{1}{(f^n)' - b}) \\
 & \quad + \frac{1}{4}T(r, f^n) + S(r, f^n) \\
 & \leq \frac{3}{4}T(r, f^n) + N(r, \frac{1}{(f^n)' - b}) + S(r, f^n)
 \end{aligned}$$

Thus, we obtain

$$T(r, f^n) = S(r, f^n).$$

Which is a contradiction, therefore the theorem is valid in this case. Now, consider the case when  $f(z)$  has infinitely many zeros  $\{z_i\}, i = 1, 2, 3, \dots$ . Define

$$g(z) = f^n(z) - bz, \text{ then } g'(z) = (f^n)'(z) - b.$$

If we show that  $g'(z)$  has infinitely many zeros then we have done. Suppose  $g'(z)$  has only finitely many zeros, so  $g(z)$  has only finitely many critical values and hence  $g(z)$  has only finitely many asymptotic values. Without any loss of generality we may assume that  $f(0) \neq \infty$ , thus by Lemma 2.10, we get

$$|g'(z_i)| \geq \frac{|g(z_i)|}{2\pi|z_i|} \log \frac{|g(z_i)|}{R},$$

this shows

$$\frac{|z_i g'(z_i)|}{|g(z_i)|} \geq \frac{1}{2\pi} \log \frac{|g(z_i)|}{R},$$

Since  $\frac{1}{2\pi} \log \frac{|g(z_i)|}{R} \rightarrow \infty$  as  $i \rightarrow \infty$ ,  $\frac{|z_i g'(z_i)|}{|g(z_i)|} \rightarrow \infty$  as  $i \rightarrow \infty$ . But  $\frac{|z_i g'(z_i)|}{|g(z_i)|} \rightarrow 1$  as  $i \rightarrow \infty$ , a contradiction. Hence we deduce that  $(f^n)'(z) - b$  has infinitely many zeros. This completes the proof of theorem. □

**Lemma 2.13.** [4] *Let  $f$  be an entire function. If the spherical derivative  $f^\#$  is bounded in  $\mathbb{C}$ , then the order of  $f$  is at most 1.*

### 3. PROOF OF THEOREM 1.1

Since normality is a local property, we assume that  $D = \Delta = \{z : |z| < 1\}$ . Suppose  $\mathcal{F}$  is not normal in  $D$ . Without loss of generality we assume that  $\mathcal{F}$  is not normal at the point  $z_0$  in  $\Delta$ . Then by Lemma 2.1, there exist

- (1) a sequence of complex numbers  $z_j \rightarrow z_0$ ,  $|z_j| < r < 1$ ,
- (2) a sequence of functions  $f_j \in \mathcal{F}$  and
- (3) a sequence of positive numbers  $\rho_j \rightarrow 0$ ,

such that  $g_j(\zeta) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j \zeta) \rightarrow g(\zeta)$  converges locally uniformly to a non-constant meromorphic function  $g(\zeta)$  in  $\mathbb{C}$  with  $g^\#(\zeta) \leq g(0) = 1$ . Moreover  $g$  is of order at most two. We see that

$$(3.1) \quad (g_j^n)^{(k)}(\zeta) = (f_j^n)^{(k)}(z_j + \rho_j \zeta) \rightarrow (g^n)^{(k)}(\zeta)$$

converges locally uniformly with respect to the spherical metric. By Hurwitz's Theorem,  $(g^n)^{(k)} \equiv 1$  or  $(g^n)^{(k)} \neq 1$ .

Let  $(g^n)^{(k)} \equiv 1$ , Then  $g$  has no pole this implies that  $g$  is an entire function having no zero. Since  $g^\# \leq 1$ , we may put  $g(\zeta) = \exp(c\zeta + d)$ , where  $c(\neq 0)$  and  $d$  are constants. therefore we get

$$(nc)^k \exp(c\zeta + d) \equiv 1,$$

which is not possible.

Thus  $(g^n)^{(k)} \neq 1$ , which contradicts Lemma 2.4 and Lemma 2.12. Thus  $\mathcal{F}$  is normal in  $\mathcal{D}$ . This completes the proof of theorem.

## 4. PROOF OF THEOREM 1.2

Since normality is a local property, we assume that  $D = \Delta = \{z : |z| < 1\}$ . Suppose  $\mathcal{F}$  is not normal in  $D$ . Without loss of generality we assume that  $\mathcal{F}$  is not normal at the point  $z_0$  in  $\Delta$ . Then by Lemma 2.1, there exist

- (1) a sequence of complex numbers  $z_j \rightarrow z_0$ ,  $|z_j| < r < 1$ ,
- (2) a sequence of functions  $f_j \in \mathcal{F}$  and
- (3) a sequence of positive numbers  $\rho_j \rightarrow 0$ ,

such that  $g_j(\zeta) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j \zeta) \rightarrow g(\zeta)$  converges locally uniformly to a non-constant meromorphic function  $g(\zeta)$  in  $\mathbb{C}$  with  $g^\#(\zeta) \leq g(0) = 1$ . Moreover  $g$  is of order at most two. We see that

$$(4.1) \quad (g_j^n)^{(k)}(\zeta) = (f_j^n)^{(k)}(z_j + \rho_j \zeta) \rightarrow (g^n)^{(k)}(\zeta)$$

converges locally uniformly with respect to the spherical metric.

Now we claim  $(g_j^n)^{(k)} - 1$  has at most one zero IM. Suppose  $(g_j^n)^{(k)} - 1$  has two distinct zeros  $\zeta_0$  and  $\zeta_0^*$  and choose  $\delta > 0$  small enough so that  $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$  and  $(g_j^n)^{(k)} - 1$  has no other zeros in  $D(\zeta_0, \delta) \cup D(\zeta_0^*, \delta)$ , where  $D(\zeta_0, \delta) = \{\zeta : |\zeta - \zeta_0| < \delta\}$  and  $D(\zeta_0^*, \delta) = \{\zeta : |\zeta - \zeta_0^*| < \delta\}$ . By Hurwitz's theorem, there exist two sequences  $\{\zeta_j\} \subset D(\zeta_0, \delta)$ ,  $\{\zeta_j^*\} \subset D(\zeta_0^*, \delta)$  converging to  $\zeta_0$ , and  $\zeta_0^*$  respectively and from (4.1), for sufficiently large  $j$ , we have

$$(f_j^n)^{(k)}(z_j + \rho_j \zeta_j) - 1 = 0 \text{ and } (f_j^n)^{(k)}(z_j + \rho_j \zeta_j^*) - 1 = 0.$$

Since  $z_j \rightarrow 0$  and  $\rho_j \rightarrow 0$ , we have  $z_j + \rho_j \zeta_j \in D(\zeta_0, \delta)$  and  $z_j + \rho_j \zeta_j^* \in D(\zeta_0^*, \delta)$  for sufficiently large  $j$ , so  $(f_j^n)^{(k)} - 1$  has two distinct zeros, which contradicts the fact that  $(f_j^n)^{(k)} - 1$  has at most one zero. But Lemma 2.5 and Lemma 2.12 confirms the non existence of such non-constant meromorphic function. This contradiction shows that  $\mathcal{F}$  is normal in  $\Delta$  and this proves the theorem.

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